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Nonlinear integral equations for complex affine Toda models associated with simply laced Lie algebras

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Abstract. A set of coupled nonlinear integral equations (NLIE) is derived for a class of models connected to the quantum group $U_q(\hat{\mathfrak{g}})$ ($\mathfrak{g}$ simply laced Lie algebra), which are solvable using the Bethe ansatz; these equations describe arbitrary excited states of a system with finite spatial length L. They generalize the simpler NLIE of the sine-Gordon/massive Thirring model to affine Toda field theory with imaginary coupling constant. As an application, the central charge and all the conformal weights of the UV conformal field theory are extracted in a straightforward manner. The quantum group truncation for q at a root of unity is discussed in detail; in the UV limit we recover through this procedure the RCFTs with extended $W(\mathfrak{g})$ conformal symmetry.

1. Introduction

The study of finite-size effects in two-dimensional (2D) solvable lattice models (SLM) or integrable quantum field theories (IQFT) has proven to be a useful tool to probe the physics of such systems. For critical SLM or conformal field theories, one can extract from finite-size corrections the central charge as well as all conformal weights [1]. For IQFT which have non-trivial renormalization group flow (including massive theories), one can obtain information on the vicinity of the UV fixed point, of the IR fixed point, and the crossover between the two.

One possible way to investigate finite-size effects is using the thermodynamic Bethe ansatz (TBA) equations [2] which describe IQFT at finite temperature T. This amounts to considering the Euclidean theory on a cylinder with radius $\beta = 1/T$. Modular invariance implies that this can also be considered as the same theory at zero temperature, but on a space which has been compactified with finite length $L = \beta$. In this formulation, the TBA equations only describe the ground state of the theory; in particular, in the limit $T \to \infty$, one can extract the central charge but not the conformal weights of the UV conformal field theory, which correspond to low-lying excited states.

Usually, in the TBA approach, the central charge appears under the form of a dilogarithmic sum, and it is known that one can extend these dilogarithmic computations to obtain all conformal weights. So it is clear that the TBA, after appropriate modification, may also yield excited states [3].

Here another approach is used, which is related to the methods of [4] and generalizes independant work by Destri and De Vega [5, 6] The idea is to study directly the Bethe

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ansatz equations for an arbitrary state, on a space of finite length L. The Bethe ansatz are then replaced with nonlinear integral equations (NLIE) which are much easier to handle; in particular they can be solved numerically. All relevant information on the system, including its energy, can then be extracted from the NLIE.

It is not clear at present how to write down NLIE for an arbitrary integrable theory. The model investigated here is the generalization of the massive Thirring/sine-Gordon model, which has $U_q(\widehat{\mathfrak{sl}(2)})$ symmetry in the $L \to \infty$ (infinite space) limit: we replace $\mathfrak{sl}(2)$ with an arbitrary simply laced Lie algebra $\mathfrak{g} = A_n, D_n, E_{6,7,8}$ and consider the associated untwisted affine Lie algebra $\hat{\mathfrak{g}}$. The continuous theory, which is obtained as the scaling limit of an inhomogenenous SLM—the fact that we consider inhomogeneous transfer matrices ensures the appearance of a mass gap in the theory—is conjectured to be the affine Toda field theory $A_n^{(1)}$, $D_n^{(1)}$ or $E_{6,7,8}^{(1)}$ with imaginary coupling constant. We shall give some strong arguments in favour of this hypothesis (mass spectrum and scattering compatible with what has been conjectured before, correct UV limit). We shall first write the standard Bethe ansatz equations using the algebraic Bethe ansatz, then transform them into the NLIE, then finally take the scaling limit. One should note that this scaling limit is not the same as the one which leads to the TBA equations. The TBA procedure involves two stages: first sending the spatial size L to infinity in such a way that the density of Bethe ansatz roots and the inverse lattice spacing remain of the order of the physical energy scale (which will eventually be the temperature T); then the inverse lattice spacing becomes the UV cut-off of the theory and is sent to infinity keeping the mass scale m of the order of T (m/T fixed). In contrast, here we shall keep L finite: it will precisely define the energy scale (there is no temperature T, since we are considering the ground state and the low-lying excited states of the theory). Thus, there is only one stage, which is to send the UV cut-off (the inverse lattice spacing) to infinity while keeping mL fixed.

As expected, for an algebra $\mathfrak{g}$ of rank n, the NLIE form a set of n coupled equations labelled by a Dynkin diagram index. However, the structure is still much simpler than the corresponding TBA equations, which, owing to the 'string hypothesis', are labelled by a second string index. One consequence of the simple structure of the NLIE is that the UV central charge and conformal weights will not appear as infinite dilogarithmic sums (as in the TBA), but as elementary finite sums.

One of the reasons which make affine Toda field theories with imaginary coupling interesting 2D integrable field theories is that, in spite of the non-Hermitianness of its Hamiltonian [7] it shares many properties of the sine-Gordon theory: solitonic excitations (which are expected to form $U_q(\mathfrak{g})$ multiplets at the quantum level, see [8]), breathers in the attractive regime [9]. It is also expected that one can consistently restrict these theories at rational values of γ/π ($q = -\mathrm{e}^{-\mathrm{i}\gamma}$) to yield (possibly unitary) theories which are perturbations of $W(\mathfrak{g})$-symmetric rational conformal field theories (RCFTs) [10]. All these points are discussed in this paper.

This paper is organized as follows. In section 2, we review some basic facts about the relevant lattice models and their Bethe ansatz equations. In section 3, we study these equations and turn them into the NLIE. Sections 4 and 5 are devoted to the computation of the energy/momentum and of the $\mathfrak{g}_0$ weight (which corresponds to the $U_q(\mathfrak{g})$ representation in the $L \to \infty$ limit) of a Bethe ansatz state. In section 6 we briefly discuss the $L \to \infty$ limit, whereas sections 7 and 8 are related to the $L \to 0$ limit and the computation of the UV central charge/conformal weights. Section 8 explains how to perform the quantum group truncation in the Bethe ansatz formalism. This leads to the consideration of a restricted theory which is shown to be a perturbation of $W(\mathfrak{g})$-symmetric minimal models by studying its UV limit. Finally, appendix A clarifies the intepretation of the UV spectrum, explaining some technical issues which were previously unclear even in the A_1 case.

2. The lattice model and its Bethe ansatz equations

Let $\mathfrak{g}$ be a simply laced Lie algebra, and $\hat{\mathfrak{g}}$ the corresponding untwisted affine algebra. We start with the R-matrix associated with $U_q(\hat{\mathfrak{g}})$ and the fundamental representation V of $\mathfrak{g}$. For $\mathfrak{g} = A_n$, we have $(a, b = 1 \dots n+1, a \neq b)$

$$\begin{aligned} \check{R}^{aa}_{aa}(\Lambda) &= 1 \\ \check{R}^{ab}_{ba}(\Lambda) &= \frac{\sin \Lambda}{\sin(\gamma - \Lambda)} \\ \check{R}^{ab}_{ab}(\Lambda) &= \frac{\sin \gamma}{\sin(\gamma - \Lambda)} \mathrm{e}^{\mathrm{i}\Lambda \operatorname{sign}(a-b)}. \end{aligned} \tag{2.1}$$

R-matrices for other Lie algebras may be found in [11]. γ is the anisotropy parameter (which is related to the deformation parameter q of $U_q(\hat{\mathfrak{g}})$ by $q = -\mathrm{e}^{-\mathrm{i}\gamma}$). We define next the inhomogeneous transfer matrix $T(\Lambda, \Theta)$; it is an operator in the physical Hilbert space $\mathcal{H} = V^{\otimes 2M}$ and it depends on a spectral parameter Λ and an inhomogeneity Θ (which will eventually play the role of UV cut-off in spectral parameter space)

$$T(\Lambda, \Theta) = \operatorname{tr}_{\text{aux}}[R_1(\Lambda - \mathrm{i}\Theta) R_2(\Lambda + \mathrm{i}\Theta) \dots R_{2M-1}(\Lambda - \mathrm{i}\Theta) R_{2M}(\Lambda + \mathrm{i}\Theta)]. \tag{2.2}$$

The R-matrix R_i is simply the R-matrix acting on the tensor product of the ith component of $\mathcal{H}$ and of an auxiliary space $V_{\text{aux}} \equiv V$ (with a permutation for correct labelling of the two spaces):

$$R_i(\Lambda) = \check{R}_{i,\text{aux}}(\Lambda) \mathcal{P}_{i,\text{aux}}. \tag{2.3}$$

The trace in (2.2) is taken on the auxiliary space. For $\mathfrak{g} = A_1$, one can redefine the Boltzmann weights (2.1) to make them real, and (2.2) reduces to the transfer matrix of the six-vertex model, with anisotropy γ. Let us also mention that if one removed the inhomogeneity Θ, then one could also describe $T(\Lambda)$ as the generating function for commuting Hamiltonians in the XXZ model and its generalizations to higher rank algebras.

The diagonalization of $T(\Lambda, \Theta)$ leads to the so-called algebraic Bethe ansatz [12]. First, one notes that $T(\Lambda, \Theta)$ commutes with the natural action of the commutative Cartan algebra $\mathfrak{g}_0$ on $\mathcal{H}$; therefore all eigenstates can be chosen weight vectors, i.e. eigenvectors of $\mathfrak{g}_0$.

The eigenstates of T are created from the highest weight vector of $\mathcal{H}$ by the action of lowering operators which are interpreted as creation operators of spin excitations. The latter are labelled by a Dynkin diagram index $s = 1 \dots n$ (see figure 1) and by a spectral parameter (which we rescale by a factor $\mathrm{i}\gamma h/2\pi$ and shall now call *rapidity*) $\lambda_{s,k}$ (where $k = 1 \dots M_s$ runs over all spin excitations of type s). As the model we consider is in an 'antiferromagnetic' regime, these excitations are not yet the physical excitations of the system. Their rapidities satisfy the following set of coupled algebraic equations (nested Bethe ansatz equations):

$$\begin{aligned} &\prod_{\substack{j=1 \\ j \neq k}}^{M_s} \frac{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \lambda_{s,j}) + \mathrm{i}))}{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \lambda_{s,j}) - \mathrm{i}))} \prod_{t|\langle st \rangle} \prod_{j=1}^{M_t} \frac{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \lambda_{t,j}) - \mathrm{i}/2))}{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \lambda_{s,j}) + \mathrm{i}/2))} \\ &\qquad = \left[\frac{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \theta) + \mathrm{i}/2))}{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \theta) - \mathrm{i}/2))} \frac{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} + \theta) + \mathrm{i}/2))}{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} + \theta) - \mathrm{i}/2))} \right]^{M\delta_{s1}} \end{aligned} \tag{2.4}$$

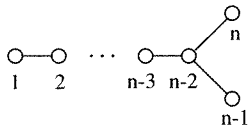
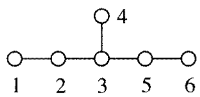
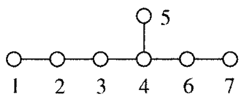
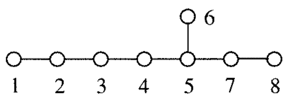
Lie algebra $\mathfrak{g}$	Dynkin diagram	Coxeter number h	$\dim \mathfrak{g}$
A_n		$n+1$	$n(n+2)$
D_n		$2(n-1)$	$n(2n-1)$
E_6		12	78
E_7		18	133
E_8		30	248

Figure 1. Table of simply laced Lie algebras. Note that $\dim \mathfrak{g} = n(h+1)$.

where $\langle st \rangle$ means that s and t are neighbours on the Dynkin diagram of $\mathfrak{g}$. Θ has also been rescaled: $\Theta \equiv \gamma h \theta / 2\pi$. To each set of $\{\lambda_{s,k}\}$ corresponds a Bethe ansatz state; calling $\mathrm{e}^{-\mathrm{i}E_\pm}$ the eigenvalue of T for $\lambda = \pm\theta$, we have

$$\mathrm{e}^{-\mathrm{i}E_\pm} = \prod_{k=1}^{M_1} \frac{\sinh(\gamma(\frac{h}{2\pi}(\mp\lambda_{1,k} + \theta) + \mathrm{i}/2))}{\sinh(\gamma(\frac{h}{2\pi}(\pm\lambda_{1,k} - \theta) + \mathrm{i}/2))}. \tag{2.5}$$

We are only interested in these particular values of λ since in the scaling limit (that will be defined later), the energy E and momentum P can be extracted from them, through the relation $E^\pm = (E \pm P)/2$, where E^+ and E^- are given by (2.5) in inverse lattice spacing (M/L) units. This relation can be derived in the 'light-cone approach' [5].

Finally, the weight r of the state with respect to $\mathfrak{g}_0$ which can be decomposed as $r = \sum_{s=1}^n r_s w_s$ on the basis of fundamental weights w_s is given by

$$r_s = \delta_{s1} 2M - \sum_{t=1}^{n} C_{st} M_t \tag{2.6}$$

where C_{st} is the Cartan matrix of $\mathfrak{g}$: $C_{st} = 2$ for $s = t$, -1 for $\langle st \rangle$, 0 otherwise.

For $\mathfrak{g} = A_n$, r_s is simply interpreted as the number of columns of size s in the Young tableau corresponding to r.

3. Derivation of the NLIE

We shall now study solutions of the Bethe ansatz equations (2.4) which correspond to low-lying excited states (i.e. states with a finite number of physical excitations above the vacuum); we shall derive for each such solution a set of NLIE, and then take the scaling limit.

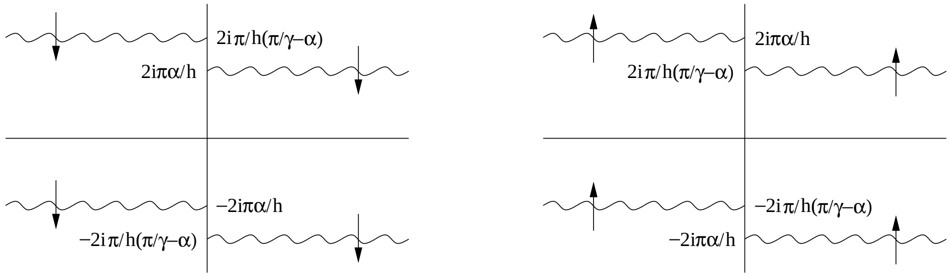


Figure 2. Cuts of the function $\phi_\alpha(\lambda)$ for $\alpha < \frac{\pi}{2\gamma}$ (left) and $\alpha > \frac{\pi}{2\gamma}$ (right). Arrows correspond to jumps of $+2\pi$ of the function.

3.1. The counting functions

The basic quantities we need are the counting functions Z_s, $s = 1 \dots n$, defined by

$$Z_s(\lambda) = \delta_{s1} M(\phi_{1/2}(\lambda+\theta) + \phi_{1/2}(\lambda-\theta)) - \sum_{k=1}^{M_s} \phi_1(\lambda - \lambda_{s,k}) + \sum_{t|\langle st\rangle} \sum_{k=1}^{M_t} \phi_{1/2}(\lambda - \lambda_{t,k}) \tag{3.1}$$

where we have introduced the notation

$$\phi_\alpha(\lambda) \equiv \mathrm{i} \log \frac{\sinh(\gamma(+\frac{h}{2\pi}\lambda + \mathrm{i}\alpha))}{\sinh(\gamma(-\frac{h}{2\pi}\lambda + \mathrm{i}\alpha))}. \tag{3.2}$$

The odd functions ϕ_α are extended to the whole complex plane by giving a prescription on their cuts (see figure 2; this is the same convention as in [6]).

The key property of Z_s is that, according to the Bethe ansatz equation (2.4), for each root $\lambda_{s,k}$ of the Bethe ansatz equation, we have

$$Z_s(\lambda_{s,k}) = 2\pi I_{s,k} \tag{3.3}$$

where one can check that $I_{s,k}$ is a half-integer, whose parity (i.e. $2I_{s,k}$ mod 2) is the opposite of that of $r_s + M_s$.

It should be noted that property (3.3) is true not only for real roots but also for complex roots (since Z_s has been defined on the whole complex plane), in contrast to what is usually done when writing Bethe ansatz equations in the thermodynamic limit.

Now let us classify the different types of roots that appear; we shall restrict ourselves to configurations of roots which survive in the scaling limit.

• *Real roots and holes.*

Since it is known that the ground state consists of real roots of all types ($s = 1 \dots n$), we expect that for an arbitrary low-lying excited state, we shall have a large number of real roots (divergent in the thermodynamic limit) that we denote by $\rho_{s,k}$ ($k = 1 \dots M_{R,s}$). If one considered the ground state, the real roots would in fact exhaust all the half-integer values (with appropriate parity) of $Z_s(\lambda)/2\pi$ with λ real; for an excited state, on the other hand, there may be a finite number of real λ which are distinct from all $\rho_{s,k}$ but still satisfy this property. We call these holes and write

$$Z_s(\eta_{s,k}) = 2\pi I_{H,s,k} \qquad k = 1 \dots M_{H,s}. \tag{3.4}$$

• *Special roots/holes.*

Because of the driving term $\delta_{s1} M(\phi_{1/2}(\lambda+\theta) + \phi_{1/2}(\lambda-\theta))$, which acts on equation $s = 1$ and is transmitted to all equations by the nearest-neighbour interaction on the Dynkin diagram, it is clear that Z_s must in general be an increasing function on the real axis. In fact for a thermodynamic state (for example at finite temperature, when the number of excitations is large) this statement is certainly true. However, it was pointed out in [6] that for low-lying excited states, there might be local variations of Z_s wing to isolated roots, so that Z_s is decreasing on a small interval. This behaviour may become important if Z_s decreases enough to cross again 2π times a half-integer. We therefore introduce real parameters $\sigma_{s,k}$ satisfying

$$Z_s(\sigma_{s,k}) = 2\pi I_{S,s,k} \qquad k = 1 \ldots M_{S,s} \tag{3.5}$$

and $Z'_s(\sigma_{s,k}) < 0$. The $\sigma_{s,k}$ may or may not be roots of the Bethe ansatz equations: they are called respectively special roots and special holes.

• *Complex roots.*

Because of the $2\pi^2/h\gamma$-periodicity of the equations, one can assume that $|\operatorname{Im}\lambda_{s,k}| \leqslant \pi^2/h\gamma$. With this convention, all non-real roots of the Bethe ansatz equations will be called complex roots and denoted by $\xi_{s,k}$, $k = 1 \ldots M_{C,s}$. Further distinctions must be introduced to classify the complex roots. We shall treat together the two regimes: repulsive regime for $\gamma < \pi/2$ and attractive regime for $\gamma > \pi/2$.

The first classification is the following: $\xi_{s,k}$ is called a wide root if $|\operatorname{Im}\xi_{s,k}| > \min(2\pi/h, 2\pi/h(\pi/\gamma - 1))$, a close root otherwise. There are $M_{\mathrm{wide},s}$ wide roots of type s and $M_{\mathrm{close},s}$ close roots.

We also define a second, independent classification: $\xi_{s,k}$ is called of the first kind if $|\operatorname{Im}\xi_{s,k}| > \pi/h$, of the second kind otherwise. We call $M_{C1,s}$ (resp. $M_{C2,s}$) the numbers of roots of the first (resp. second) kind.

To clarify these definitions, we notice that there are three cases depending on the value of γ. In the repulsive regime ($\gamma < \pi/2$), all wide roots are of the first kind, whereas a close root ξ can be either of the first kind ($\pi/h < |\operatorname{Im}\xi| < 2\pi/h$) or of the second kind ($0 < |\operatorname{Im}\xi| < \pi/h$). In the 'weakly attractive' regime ($\pi/2 < \gamma < 2\pi/3$), again all wide roots are of the first kind, and there are still close roots of the first kind ($\pi/h < |\operatorname{Im}\xi| < 2\pi/h(\pi/\gamma - 1)$). Finally, in the 'strongly attractive' regime, all close roots are of the second kind, and wide roots are either of the second kind ($2\pi/h(\pi/\gamma - 1) < |\operatorname{Im}\xi| < \pi/h$) or of the first kind ($|\operatorname{Im}\xi| > \pi/h$).

3.2. The NLIE

The derivation is a straightforward generalization of [6]. We first assume that there are no special roots/holes. We then use the following trick: as the real zeros of the function $1 + (-1)^{\delta_s} \mathrm{e}^{\mathrm{i}Z_s(z)}$ ($\delta_s \equiv r_s + M_s \bmod 2$) are exactly the real roots and the holes of type s, we have

$$\oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} f(z) \frac{\mathrm{d}}{\mathrm{d}z} \log(1 + (-1)^{\delta_s} \mathrm{e}^{\mathrm{i}Z_s(z)}) = \sum_{k=1}^{M_{R,s}} f(\rho_{s,k}) + \sum_{k=1}^{M_{H,s}} f(\eta_{s,k}) \tag{3.6}$$

for an arbitrary analytic function f. The contour C is a closed curve which encircles all the $\rho_{s,k}$ and $\eta_{s,k}$.

We apply (3.6) to the definition (3.1) of $Z_s(\lambda)$ (λ real) differentiated once: we obtain

$$Z'_s(\lambda) = \delta_{s1} 2\pi M(\Phi_{1/2}(\lambda + \theta) + \Phi_{1/2}(\lambda - \theta))$$
$$-\bigg[\oint_C \frac{\mathrm{d}z}{\mathrm{i}} \Phi_1(\lambda - z) \frac{\mathrm{d}}{\mathrm{d}z} \log(1 + (-1)^{\delta_s} \mathrm{e}^{\mathrm{i}Z_s(z)})$$

$$
-2\pi \sum_{k=1}^{M_{H,s}} \Phi_1(\lambda - \eta_{s,k}) + 2\pi \sum_{k=1}^{M_{C,s}} \Phi_1(\lambda - \xi_{s,k}) \Bigg]
$$
$$
+ \sum_{t|\langle st\rangle} \Bigg[\oint_C \frac{\mathrm{d}z}{\mathrm{i}} \Phi_{1/2}(\lambda - z) \frac{\mathrm{d}}{\mathrm{d}z} \log(1 + (-1)^{\delta_t} \mathrm{e}^{\mathrm{i}Z_t(z)})
$$
$$
-2\pi \sum_{k=1}^{M_{H,t}} \Phi_{1/2}(\lambda - \eta_{t,k}) + 2\pi \sum_{k=1}^{M_{C,t}} \Phi_{1/2}(\lambda - \xi_{t,k}) \Bigg] \tag{3.7}
$$

where we have introduced $\Phi_\alpha = \frac{1}{2\pi}\mathrm{d}\phi_\alpha/\mathrm{d}\lambda$. Next we deform the contour C so that the contour integrals can be rewritten as integrals on the real axis:

$$
Z_s'(\lambda) = \delta_{s1} 2\pi M(\Phi_{1/2}(\lambda + \theta) + \Phi_{1/2}(\lambda - \theta)) - \Bigg[\int \mathrm{d}x\, \Phi_1(\lambda - x) Z_s'(x)
$$
$$
+ \int \mathrm{d}x\, \Phi_1(\lambda - x) \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x} \log \frac{(-1)^{\delta_s} + \mathrm{e}^{-\mathrm{i}Z_s(x-\mathrm{i}0)}}{1 + (-1)^{\delta_s} \mathrm{e}^{\mathrm{i}Z_s(x+\mathrm{i}0)}}
$$
$$
-2\pi \sum_{k=1}^{M_{H,s}} \Phi_1(\lambda - \eta_{s,k}) + 2\pi \sum_{k=1}^{M_{C,s}} \Phi_1(\lambda - \xi_{s,k}) \Bigg]
$$
$$
+ \sum_{t|\langle st\rangle} \Bigg[\int \mathrm{d}x\, \Phi_{1/2}(\lambda - x) Z_t'(x)
$$
$$
+ \int \mathrm{d}x\, \Phi_{1/2}(\lambda - x) \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x} \log \frac{(-1)^{\delta_t} + \mathrm{e}^{-\mathrm{i}Z_t(x-\mathrm{i}0)}}{1 + (-1)^{\delta_t} \mathrm{e}^{\mathrm{i}Z_t(x+\mathrm{i}0)}}
$$
$$
-2\pi \sum_{k=1}^{M_{H,t}} \Phi_{1/2}(\lambda - \eta_{t,k}) + 2\pi \sum_{k=1}^{M_{C,t}} \Phi_{1/2}(\lambda - \xi_{t,k}) \Bigg]. \tag{3.8}
$$

This equation suggests the introduction of the real function

$$
Q_s(x) = \frac{1}{\mathrm{i}} \log \frac{1 + (-1)^{\delta_s} \mathrm{e}^{\mathrm{i}Z_s(x+\mathrm{i}0)}}{1 + (-1)^{\delta_s} \mathrm{e}^{-\mathrm{i}Z_s(x-\mathrm{i}0)}} \tag{3.9}
$$

(note that Q_s is *a priori* defined up to a constant since only its derivative appears in (3.8), but the definition above turns out to be convenient). Q_s clearly satisfies

$$
Q_s(x) = (Z_s(x) + \delta_s \pi) \bmod 2\pi \tag{3.10}
$$

so that it is entirely defined on the real axis (by appropriate choice of the logarithmic cuts) by imposing (3.10) and $|Q_s(x)| \leqslant \pi$.

To simplify (3.8) we introduce the spectral-parameter dependent Cartan matrix $C_{st}(\lambda)$:

$$
C_{st}(\lambda) \equiv \begin{cases} 2\delta(\lambda) & s = t \\ -\dfrac{h}{2\pi} \dfrac{1}{\cosh(h\lambda/2)} \equiv -2s(\lambda) & \langle st\rangle \end{cases} \tag{3.11}
$$

for $1 \leqslant s, t \leqslant n$. In the following we shall also use Fourier transform defined by

$$
f(\kappa) = \int \mathrm{d}\lambda\, \exp(\mathrm{i}\kappa h\lambda/\pi) f(\lambda) \tag{3.12}
$$

for any function f. With this convention $s(\kappa) = 1/(2\cosh(\kappa))$. Note in particular that $C_{st}(\kappa = 0) \equiv C_{st}$ is the usual Cartan matrix of $\mathfrak{g}$.

We now rewrite (3.8):

$$\sum_{t=1}^{n} C_{st} \star (1+\Phi_1) \star Z_t'(\lambda) = \delta_{s1} 4\pi M(\Phi_{1/2}(\lambda+\theta)+\Phi_{1/2}(\lambda-\theta))$$
$$+\sum_{t=1}^{n}\Bigg[(C_{st}\star(1+\Phi_1)-2\delta_{st})\star\frac{\mathrm{d}}{\mathrm{d}\lambda}Q_t$$
$$-2\pi\sum_{k=1}^{M_{H,t}}(C_{st}\star(1+\Phi_1)-2\delta_{st})(\lambda-\eta_{t,k})$$
$$+2\pi\sum_{k=1}^{M_{C,t}}(C_{st}\star(1+\Phi_1)-2\delta_{st})(\lambda-\xi_{t,k})\Bigg] \qquad (3.13)$$

where $\star$ means convolution product in λ space, and 1 is the identity operator (convolution with the δ function).

We multiply by the inverse matrix $C_{st}^{-1} \star (1+\Phi_1)^{-1}$, and then take the scaling limit $M \to \infty$, $\theta \to \infty$ keeping $mL \equiv M\mathrm{e}^{-\theta}$ fixed. Expanding the inhomogeneous term $4\pi C_{s1}^{-1} \star (s(\lambda+\theta)+s(\lambda-\theta))$ as $\theta \to \infty$, one finds [13] that the Perron–Frobenius eigenvalue of the Cartan matrix (or, more precisely, of the adjacency matrix of its Dynkin diagram) dominates, so that

$$Z_s' = m_s L\cosh\lambda + \sum_{t=1}^{n}\left[X_{st}\star\frac{\mathrm{d}}{\mathrm{d}\lambda}Q_t + \sum_{k=1}^{M_{H,t}} X_{st}(\lambda-\eta_{t,k}) - \sum_{k=1}^{M_{C,t}} X_{st}(\lambda-\xi_{t,k})\right]. \qquad (3.14)$$

The m_s are the masses of the solitons of the theory; they form the Perron–Frobenius eigenvector, and they are of the order of the mass scale m. The X_{st} are regular functions; on the real axis they are given by $X_{st}(\lambda) = \delta_{st}\delta(\lambda) - 2(1+\Phi_1)^{-1}\star C_{st}^{-1}$, so that their Fourier transforms are

$$X_{st}(\kappa) = \delta_{st} - \frac{\sinh(\frac{\pi}{\gamma}\kappa)}{\sinh((\frac{\pi}{\gamma}-1)\kappa)\cosh(\kappa)} C_{st}^{-1}(\kappa) \qquad (3.15)$$

where $C_{st}^{-1}(\kappa)$ is listed in figure 3 for the infinite series A_n, D_n.

However, to define $X_{st}(\lambda-\xi_{t,k})$ (3.14) one must extend X_{st} to the complex plane; one must then be careful that the poles of the functions Φ_1 and $\Phi_{1/2}$ are smeared by the convolution product with $C_{st}^{-1}\star(1+\Phi_1)^{-1}$ and become cuts running parallel to the real axis. In other words, X_{st} is not simply the analytic continuation $X_{st}^{(0)}$ of its definition on the real axis; rather, it is given by

$$X_{st}(\lambda) = X_{st}^{(0)}(\lambda) + \vartheta\left(\operatorname{Im}\lambda - \frac{2\pi}{h}\right)X_{st}^{(0)}\left(\lambda - \mathrm{i}\frac{2\pi}{h}\right)$$
$$-\vartheta\left(\operatorname{Im}\lambda - \frac{2\pi}{h}(\pi/\gamma-1)\right)X_{st}^{(0)}\left(\lambda-\mathrm{i}\frac{2\pi}{h}(\pi/\gamma-1)\right)$$
$$-\vartheta\left(\operatorname{Im}\lambda-\frac{\pi}{h}\right)\sum_{t'|\langle tt'\rangle} X_{st'}^{(0)}\left(\lambda-\mathrm{i}\frac{\pi}{h}\right) \qquad (3.16)$$

for $\operatorname{Im}\lambda > 0$ (and a similar expression for $\operatorname{Im}\lambda < 0$). ϑ is the usual step function.

Finally one can safely integrate once (3.14) (taking care of the integration constant, which vanishes since it has been absorbed in the definition (3.9) of Q_s), and re-introduce

$\mathfrak{g}$	$C_{st}^{-1}(\kappa),\ s \geqslant t$	P-F. eigenvector
A_n	$\coth(\kappa)\dfrac{\sinh((n+1-s)\kappa)\sinh(t\kappa)}{\sinh((n+1)\kappa)}$	$\sin\left(\dfrac{\pi s}{n+1}\right)$
D_n	$\begin{cases} \coth(\kappa)\dfrac{\cosh((n-1-s)\kappa)\sinh(t\kappa)}{\cosh((n-1)\kappa))} & s \leqslant n-2 \\ \coth(\kappa)\dfrac{\sinh(t\kappa)}{2\cosh((n-1)\kappa))} & s \geqslant n-1,\ t \leqslant n-2 \\ \dfrac{\sinh(n\kappa)}{2\sinh(\kappa)\cosh((n-1)\kappa)} & s = t \geqslant n-1 \\ \dfrac{\sinh((n-2)\kappa)}{2\sinh(\kappa)\cosh(n-1)\kappa)} & s = n,\ t = n-1 \end{cases}$	$\begin{cases} \sin\left(\dfrac{\pi s}{2(n-1)}\right) & s \leqslant n-2 \\ \frac{1}{2} & s = n-1,\ n \end{cases}$

Figure 3. Table of inverse Cartan matrices $C_{st}^{-1}(\kappa)$ and of the Perron–Frobenius eigenvectors (which give the mass spectrum).

the special roots/holes as in [6]. The final equation is

$$Z_s = m_s L \sinh\lambda + \sum_{t=1}^{n}\left[X_{st}\star Q_t + \sum_{k=1}^{M_{H,t}}\chi_{st}(\lambda-\eta_{t,k}) \right. \\ \left. -2\sum_{k=1}^{M_{S,t}}\chi_{st}(\lambda-\sigma_{t,k}) - \sum_{k=1}^{M_{C,t}}\chi_{st}(\lambda-\xi_{t,k})\right] \tag{3.17}$$

with χ_{st} the odd primitive of $2\pi X_{st}$ (for $\chi_{st}(\lambda-\xi_{t,k})$ one should integrate on a line parallel to the real axis).

4. Energy and momentum in the scaling limit

To each set of counting functions Z_s that satisfy the NLIE (3.17) is associated a configuration of holes and complex roots which characterizes the corresponding excited state. Conversely, specifying the approximate positions of holes and complex roots†, one can solve the nonlinear equations (3.17) (at least numerically) and obtain the counting functions $Z_s(\lambda)$. We shall now go on and show how to express the energy/momentum in terms of the Z_s. We shall not give all the details of the derivation since it is very similar to the derivation of the NLIE itself. We introduce the auxiliary function

$$W(\lambda) = \sum_{k=1}^{M_1}\Phi_{1/2}(\lambda-\lambda_{1,k}) \tag{4.1}$$

and use the contour integral trick to express it as

$$W(\lambda) = \oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}}\Phi_{1/2}(\lambda-z)\frac{\mathrm{d}}{\mathrm{d}z}\log(1+(-1)^{\delta_1}\mathrm{e}^{\mathrm{i}Z_1(z)}) - \sum_{k=1}^{M_{H,1}}\Phi_{1/2}(\lambda-\eta_{1,k})$$

† The positions of holes cannot not be chosen arbitrarily since they are 'quantized' at finite L. The quantization condition itself depends on positions of other holes and complex roots because of the interaction between the roots.

$$+2\sum_{k=1}^{M_{S,1}}\Phi_{1/2}(\lambda-\sigma_{1,k})+\sum_{k=1}^{M_{C,1}}\Phi_{1/2}(\lambda-\xi_{1,k}) \tag{4.2}$$

then use the NLIE:

$$W(\lambda)=W_0(\lambda)-\sum_{s=1}^{n}\left[\frac{1}{2\pi}G_s\star Q'_s-\sum_{k=1}^{M_{H,s}}G_s(\lambda-\eta_{s,k})\right.$$
$$\left.+2\sum_{k=1}^{M_{S,s}}G_s(\lambda-\sigma_{s,k})+\sum_{k=1}^{M_{C,s}}G_s(\lambda-\xi_{s,k})\right]. \tag{4.3}$$

$W_0(\lambda)$ is a function that plays no role and will contribute to the ground-state bulk energy. $G_s(\lambda)\equiv 2C^{-1}_{s1}\star s(\lambda)$ on the real axis, but like the X_{st} its definition differs in the complex plane. We shall not bother to write down the analogue of equation (3.16) for G_s, since we are only interested in the scaling limit (θ and $M\to\infty$), in which this discussion simplifies drastically. Indeed, the expansion of $G_s(\lambda\pm\theta)$ (and the use of the Perron–Frobenius eigenvector property $\sum_{t|\langle st\rangle}m_t=2\cos(\pi/h)m_s$) leads to $2\pi\frac{M}{L}G_s(\lambda-\theta)\sim\frac{1}{2}m_s e(\lambda)$ where

$$e(\lambda)\equiv \mathrm{e}^{\lambda}\left(1+\vartheta\left(\operatorname{Im}\lambda-\frac{2\pi}{h}\right)\mathrm{e}^{-2\mathrm{i}\pi/h}\right.$$
$$-\vartheta\left(\operatorname{Im}\lambda-\frac{2\pi}{h}(\pi/\gamma-1)\right)\mathrm{e}^{-2\mathrm{i}\pi/h(\pi/\gamma-1)}$$
$$\left.-\vartheta\left(\operatorname{Im}\lambda-\frac{\pi}{h}\right)(1+\mathrm{e}^{-2\mathrm{i}\pi/h})\right) \tag{4.4}$$

(for $\operatorname{Im}\lambda>0$).

After integrating once $W(\lambda)$ and plugging (4.3) in the definition (2.5) of the energy, we find

$$E=\sum_s m_s\left[\sum_{k=1}^{M_{H,s}}\cosh\eta_{s,k}-2\sum_{k=1}^{M_{S,s}}\cosh\sigma_{s,k}\right.$$
$$\left.-\sum_{k=1}^{M_{C,s}}\tfrac{1}{2}(e(\xi_{s,k})+e(-\xi_{s,k}))-\frac{1}{2\pi}\int\mathrm{d}\lambda\,\cosh\lambda\, Q_s(\lambda)\right] \tag{4.5}$$

where we have discarded the bulk ground-state energy. We shall now discuss in more detail the contribution of the complex roots, which we call E_C and redecompose: $E_C=E_C^{+}+E_C^{-}$. We treat separately the repulsive and attractive regimes.

When $\gamma<\pi/2$ (repulsive regime), according to (4.4), the contribution of the wide roots to $W(\lambda)$ and E vanishes. The close roots do contribute:

$$E_C^{+}=\tfrac{1}{2}\sum_{s=1}^{n}m_s\Bigg[-\sum_{\substack{k=1\\|\operatorname{Im}\xi_{s,k}|<\pi/h}}\mathrm{e}^{\xi_{s,k}}+\sum_{\substack{k=1\\ \pi/h<|\operatorname{Im}\xi_{s,k}|<2\pi/h}}\mathrm{e}^{\xi_{s,k}-2\mathrm{i}\pi/h\epsilon_{s,k}}\Bigg] \tag{4.6}$$

where $\epsilon_{s,k}\equiv\operatorname{sign}(\operatorname{Im}\xi_{s,k})$.

When $\gamma>\pi/2$ (attractive regime), all complex roots contribute; if $\pi/2<\gamma<2\pi/3$ we find

$$E_C^{+}=\tfrac{1}{2}\sum_{s=1}^{n}m_s\Bigg[-\sum_{\substack{k=1\\|\operatorname{Im}\xi_{s,k}|<\pi/h}}^{n}\mathrm{e}^{\xi_{s,k}}+\sum_{\substack{k=1\\ \pi/h<|\operatorname{Im}\xi_{s,k}|<2\pi/h(\pi/\gamma-1)}}^{n}\mathrm{e}^{\xi_{s,k}-2\mathrm{i}\pi/h\epsilon_{s,k}}$$
$$+\sum_{\substack{k=1\\|\operatorname{Im}\xi_{s,k}|>2\pi/h(\pi/\gamma-1)}}^{n}\mathrm{e}^{\xi_{s,k}}(\mathrm{e}^{-2\mathrm{i}\pi/h\epsilon_{s,k}}+\mathrm{e}^{-2\mathrm{i}\pi/h(\pi/\gamma-1)\epsilon_{s,k}})\Bigg] \tag{4.7}$$

whereas for $\gamma > 2\pi/3$ we find

$$E_C^+ = \frac{1}{2}\sum_{s=1}^{n} m_s\Bigg[-\sum_{\substack{k=1\\|\operatorname{Im}\xi_{s,k}|<2\pi/h(\pi/\gamma-1)}}^{n} \mathrm{e}^{\xi_{s,k}} + \sum_{\substack{k=1\\2\pi/h(\pi/\gamma-1)<|\operatorname{Im}\xi_{s,k}|<\pi/h}}^{n} \mathrm{e}^{\xi_{s,k}}\left(-1+\mathrm{e}^{-2\mathrm{i}\pi/h(\pi/\gamma-1)\epsilon_{s,k}}\right) + \sum_{\substack{k=1\\|\operatorname{Im}\xi_{s,k}|>\pi/h}}^{n} \mathrm{e}^{\xi_{s,k}}\left(\mathrm{e}^{-2\mathrm{i}\pi/h\epsilon_{s,k}}+\mathrm{e}^{-2\mathrm{i}\pi/h(\pi/\gamma-1)\epsilon_{s,k}}\right)\Bigg]. \tag{4.8}$$

5. Relation between numbers of holes/complex roots and representation of the state

It is convenient to express now the weight r of the low-lying excited states in terms of quantities which remain finite when we take the scaling limit; indeed, (2.6) expresses r in terms of the M_s which diverge as $M \to \infty$. Instead we shall derive now a relation between r and the numbers of holes, special roots/holes and complex roots.

We start by considering the limit $\lambda \to +\infty$ in the definition (3.1) of Z_s: we obtain

$$\begin{aligned} Z_s(+\infty) &= \delta_{s1}(\pi-\gamma)2M - (\pi-2\gamma)M_s \\ &\quad + \sum_{t|\langle st\rangle}(\pi-\gamma)M_t + 2\pi\,\mathrm{sign}(\pi-2\gamma)M_{\mathrm{wide}\downarrow,s} - 2\pi\sum_{t|\langle st\rangle} M_{C1\downarrow,t} \\ &= (\pi-\gamma)r_s + \pi M_s + 2\pi\,\mathrm{sign}(\pi-2\gamma)M_{\mathrm{wide}\downarrow,s} - 2\pi\sum_{t|\langle st\rangle} M_{C1\downarrow,t} \end{aligned} \tag{5.1}$$

and a similar expression for $Z_s(-\infty)$. The sign $\downarrow$ indicates that we are counting the number of roots ξ which satisfy $\operatorname{Im}\xi < 0$. In principle, we have $M_{\mathrm{wide}\downarrow,s} = \frac{1}{2}M_{\mathrm{wide},s}$ and $M_{C1\downarrow,s} = \frac{1}{2}M_{C1,s}$ since complex roots come in conjugate pairs; however, we do not need these relations.

Next we count the number of integer values of Z_s on the real axis. For this purpose we introduce $I_s^{\max}$ (resp. $I_s^{\min}$), which is the largest (resp. smallest) half-integer (with appropriate parity) comprised in the inverval $[Z_s(-\infty)/2\pi, Z_s(+\infty)/2\pi]$. This definition and (5.1) imply that

$$I_s^{\max} + \tfrac{1}{2} = \tfrac{1}{2}(M_s + r_s) - E\left[\frac{1}{2} + \frac{\gamma}{2\pi}r_s\right] + \mathrm{sign}(\pi-2\gamma)M_{\mathrm{wide}\downarrow,s} - \sum_{t|\langle st\rangle} M_{C1\downarrow,t} \tag{5.2}$$

and similarly

$$I_s^{\min} - \tfrac{1}{2} = -\tfrac{1}{2}(M_s + r_s) + E\left[\frac{1}{2} + \frac{\gamma}{2\pi}r_s\right] - \mathrm{sign}(\pi-2\gamma)M_{\mathrm{wide}\uparrow,s} + \sum_{t|\langle st\rangle} M_{C1\uparrow,t}. \tag{5.3}$$

Note that $I_s^{\max}$ and $I_s^{\min}$ have the correct parity (opposite of $M_s + r_s$).

Now it is recalled that for half-integer values of Z_s on the real axis, we have real roots and holes (including special roots/holes). Using the obvious relation $M_s = M_{C,s} + M_{R,s}$, we find that

$$M_{H,s} = I_s^{\max} - I_s^{\min} + 1 - M_s + M_{C,s} + 2M_{S,s}. \tag{5.4}$$

Combining (5.2)–(5.4) and (2.6), we finally have

$$r_s = M_{H,s} - 2M_{S,s} + 2E\left[\frac{1}{2} + \frac{\gamma}{2\pi}r_s\right] - M_{\mathrm{close},s} - 2\vartheta(\pi-2\gamma)M_{\mathrm{wide},s} + \sum_{t|\langle st\rangle} M_{C1,t}. \tag{5.5}$$

Once we have obtained equation (5.5), we can take the scaling limit in it. The only simplification that occurs concerns the special roots/holes. The situation is identical to that encountered in [6], so we shall not justify the following statement in detail: as $\theta \to \infty$, some special roots/holes are sent to infinity, and their number exactly cancels the $2E[\frac{1}{2} + \frac{\gamma}{2\pi} r_s]$ in (5.5). Finally, still calling $M_{S,s}$ the number of remaining special roots/holes, we have

$$r_s = M_{H,s} - 2M_{S,s} - M_{\mathrm{close},s} - 2\vartheta(\pi - 2\gamma) M_{\mathrm{wide},s} + \sum_{t|\langle st\rangle} M_{C1,t}. \tag{5.6}$$

Alhough we shall not use this simplification in the subsequent calculations, it is in fact essential for their self-consistency.

6. Large L limit

We shall only sketch the $L \to \infty$ limit, in which we should recover the usual physics of the infinite-volume system. Starting from the NLIE. (3.17) one should be able to generalize to results of [14] to all regimes and all simply laced Lie algebras.

For all values of γ, the holes correspond to relativistic physical excitations that we identify with solitons. From (3.17) and (5.5) we infer that a hole of type s with rapidity $\eta_{s,k}$ corresponds to a soliton of mass m_s and which belongs to the fundamental representation w_s of $U_q(\hat{\mathfrak{g}})$. For example, for $\mathfrak{g} = A_1$, solitons and antisolitons are put together in a $U_q(\widehat{\mathfrak{sl}(2)})$ doublet, so here the holes correspond to solitons.

For the interpretation of the complex roots, we need to extend the NLIE (3.17) over the whole complex plane. The continuation is easily accomplished if one correctly takes into account the poles one catches when deforming the integration paths; we shall not describe the whole procedure explicitly since it is very similar to what has already been done (equations (3.16), (4.4)). We shall formally write the result as

$$Z_s(\lambda) = \frac{m_s L}{2}(e(\lambda) - e(-\lambda)) + \sum_{t=1}^{n} X_{st} \star Q_t + g_s(\lambda) \tag{6.1}$$

where for real λ,

$$g_s(\lambda) \equiv \sum_{t=1}^{n}\left[\sum_{k=1}^{M_{H,t}} \chi_{st}(\lambda - \eta_{t,k}) - 2\sum_{k=1}^{M_{S,t}} \chi_{st}(\lambda - \sigma_{t,k}) - \sum_{k=1}^{M_{C,t}} \chi_{st}(\lambda - \xi_{t,k})\right]. \tag{6.2}$$

We then impose the relation $\exp(\mathrm{i}Z_s(\xi_{s,k})) = (-1)^{1+\delta_s}$ for all complex roots $\xi_{s,k}$. The divergent imaginary part of $mL(e(\xi_{s,k}) - e(-\xi_{s,k}))$ has to be compensated for by a pole in $\exp(g_s(\xi_{s,k}))$; this forces the complex roots to fall into certain configurations.

In the repulsive case ($\gamma < \pi/2$), the close roots group into *quartets* which consists of two roots of the first kind ξ, $\bar{\xi}$ ($\mathrm{Im}\,\xi > 0$) and two roots of the second kind $\xi - 2\mathrm{i}\pi/h$, $\bar{\xi} + 2\mathrm{i}\pi/h$. This configuration can degenerate into a *2-string* ξ, $\bar{\xi}$, with $\mathrm{Im}\,\xi = \mathrm{i}\pi/h$. The contribution of the different members of the quartet to the energy exactly cancels, so that quartets have zero energy. Wide roots do not have any constraints on their rapidities since $e(\xi) = 0$ for a wide root; for the same reason they do not contribute to the energy.

The interpretation of this result is that complex roots serve as a way of lowering the weight r of the system without changing its energy. Note that close roots and wide roots do not modify r in the same way: we can rewrite (5.6) using our knowledge of the configurations of close roots

$$r_s = M_{H,s} - 2M_{S,s} - \tfrac{1}{2}\sum_{t} C_{st} M_{\mathrm{close},t} - 2M_{\mathrm{wide},s}. \tag{6.3}$$

The fact that the energy is unchanged when adding complex roots in a system with fixed holes is the sign of an enlarged symmetry at $L \to \infty$ (quantum *affine* symmetry $U_q(\hat{\mathfrak{g}})$).

The attractive regime is more complicated. Here we shall make some general observations. Equations (4.7) and (4.8) show that wide roots now carry energy: their presence is related to the appearance of *breathers*. By definition we call breathers all the particles of the spectrum which are not the fundamental solitons. In the repulsive regime, the fundamental solitons form no other bound states than themselves; but in the attractive regime, new bound states are created. According to (5.6), breathers are necessarily neutral for $\pi/2 < \gamma < 2\pi/3$, whereas they can be charged for $\gamma > 2\pi/3$. A more detailed description of the allowed configurations will be given in a forthcoming paper.

Finally, interpreting the NLIE as equations for phase shifts of physical particles on the periodic space of length L, one immediately identifies $\chi_{st}(\lambda)$ with the phase shift between two solitons of type s and t with rapidity difference λ. More precisely, this corresponds to scattering in the highest weight in the tensor product (insertion of complex roots allows us to obtain the lower weights). For example, for $\mathfrak{g} = A_n$, one has (using the expression for $X_{11}(\kappa)$ given in (3.15))

$$S_{11}(\lambda) = \exp\left(\mathrm{i}\int_0^{+\infty} \mathrm{d}\kappa\, \frac{2\sin(\kappa h\lambda/\pi)}{\kappa} \frac{\sinh((\pi/\gamma - h)\kappa)\sinh(\kappa)}{\sinh((\pi/\gamma - 1)\kappa)\sinh(h\kappa)}\right) \tag{6.4}$$

(up to a global phase). For $n = 1$ this reproduces the well known sine-Gordon soliton-soliton S-matrix. For $n > 1$ it is precisely the S-matrix conjectured in [15] for the affine Toda with imaginary coupling.

7. Large θ limit (decoupling of the two chiralities)

In preparation for the UV (conformal) limit $L \to 0$, we shall first consider the limit $\theta \to \infty$, with M large but finite. Intuitively, since $mL = M\mathrm{e}^{-\theta}$, this is basically the same as the limit $L \to 0$. Indeed, one can check that there is proper commutation of the limits, so that the results we shall obtain in this section will be valid in the next, in which we take $L \to 0$ *after* the scaling limit $M \to \infty$, $\theta \to \infty$. The advantage of keeping M finite is that just as in section 5, one can write intermediate equations which would diverge as $M \to \infty$.

In the large θ limit, the NLIE (just like the TBA equations) exhibit decoupling of the two chiralities. The functions $Z_s(\lambda)$ have a growing flat plateau in the region $[-\theta, \theta]$, which implies that if we consider the positions of roots and holes varying continuously with θ, then the set of roots and holes divides into left-movers and right-movers, according to

$$\begin{aligned} \lambda_{s,k} &= \lambda^{\pm}_{s,k} \pm \theta \\ \eta_{s,k} &= \eta^{\pm}_{s,k} \pm \theta \end{aligned} \tag{7.1}$$

where the $\lambda^{\pm}_{s,k}$ and $\eta^{\pm}_{s,k}$ ($k = 1 \ldots M^{\pm}_s, M^{\pm}_{H,s}$ after reordering of the indices) are kept fixed as $\theta \to \infty$. In particular for complex roots and special roots/holes we define the $\xi^{\pm}_{s,k}$ and $\sigma^{\pm}_{s,k}$ ($k = 1 \ldots M^{\pm}_{C,s}, M^{\pm}_{S,s}$). We allow exceptional unmoving roots or holes which may appear in special configurations, even though they will play no role for us; all this means is that we do not impose, for the moment, relations such as $M^{+}_{H,s} + M^{-}_{H,s} = M_{H,s}$. We also define for future use $r^{\pm}_s \equiv M\delta_{s1} - \sum_{t=1}^{n} C_{st} M^{\pm}_t$, in analogy with the corresponding expression for r_s.

The next step is to define the two decoupled counting functions $Z^{\pm}_s$ as

$$Z^{\pm}_s(\lambda) = \lim_{\theta \to +\infty} Z_s(\lambda \pm \theta). \tag{7.2}$$

In the intermediate region, $Z_s(\lambda)$ is flat, so that $Z_s^+(-\infty) = Z_s^-(+\infty)$, except if there are unmoving roots. To see this more clearly, let us pick $Z_s^+(-\infty)$, and compute it mod 2π. The relations we find will be useful in the calculation of the UV conformal weights.

From the definition (3.1) of Z_s, separating right-movers from the other roots, we have

$$\begin{aligned}
Z_s^+(-\infty) = &-(\pi - 2\gamma)(M_s - 2M_s^+) + \sum_{t|\langle st\rangle}(\pi - \gamma)(M_t - 2M_t^+) \\
&+2\pi\,\mathrm{sign}(\pi - 2\gamma)(M_{\mathrm{wide}\downarrow,s} - M^+_{\mathrm{wide},s}) - 2\pi\sum_{t|\langle st\rangle}(M_{C1\downarrow,t} - M^+_{C1,t}) \\
= &\,(\pi - \gamma)(r_s - 2r_s^+) + \pi(M_s - 2M_s^+) \\
&+2\pi\,\mathrm{sign}(\pi - 2\gamma)(M_{\mathrm{wide}\downarrow,s} - M^+_{\mathrm{wide},s}) - 2\pi\sum_{t|\langle st\rangle}(M_{C1\downarrow,t} - M^+_{C1,t}).
\end{aligned} \tag{7.3}$$

We now introduce $z_s^+ \equiv Q_s^+(-\infty)$, so that

$$\begin{aligned}
z_s^+ = &\,Z_s^+(-\infty) + \pi\delta_s \bmod 2\pi \\
&= Z_s^+(-\infty) - 2\pi(I_s^{\min +} - \tfrac{1}{2}) \\
&= Z_s^+(-\infty) - 2\pi(I_s^{\max} + \tfrac{1}{2} - (M^+_{H,s} - 2M^+_{S,s} + M_s^+ - M^+_{C,s})) \\
&= -2(\pi - \gamma)r_s^+ + 2\pi(M^+_{H,s} - 2M^+_{S,s}) - \left(\gamma r_s - 2\pi E\left[\frac{1}{2} + \frac{\gamma}{2\pi}r_s\right]\right) \\
&\quad -2\pi(M^+_{\mathrm{close},s} + 2\vartheta(\pi - 2\gamma)M^+_{\mathrm{wide},s}) + 2\pi\sum_{t|\langle st\rangle}M^+_{C1,t}.
\end{aligned} \tag{7.4}$$

Here, we have introduced $I_s^{\min +}$, the smallest half-integer (with appropriate parity) larger than $Z_s^+(-\infty)$, and related it to $I_s^{\max}$ in the obvious way; and we have replaced $Z_s^+(-\infty)$ with its value (7.3).

Therefore we are led to the particularly simple expression

$$z_s^+ = \gamma(2r_s^+ - r_s) + 2\pi(\hat{r}_s^+ - r_s^+) \tag{7.5}$$

with

$$\hat{r}_s^+ \equiv M^+_{H,s} - 2M^+_{S,s} + E\left[\frac{1}{2} + \frac{\gamma}{2\pi}r_s\right] - M^+_{\mathrm{close},s} - 2\vartheta(\pi - 2\gamma)M^+_{\mathrm{wide},s} + \sum_{t|\langle st\rangle}M^+_{C1,t}. \tag{7.6}$$

Comparing (7.5) with (5.5), one can intepret $\hat{r}_s^+$ as the partial quantum number induced by the right-movers. Of course, in the scaling limit, the term $E[\frac{1}{2} + \frac{\gamma}{2\pi}r_s]$ is cancelled by the extremal special roots/holes, and we can simply remove it in (7.6) (cf (5.6)).

A similar expression may be found for $z_s^- \equiv Q_s^-(+\infty)$:

$$z_s^- = -\gamma(2r_s^- - r_s) - 2\pi(\hat{r}_s^- - r_s^-). \tag{7.7}$$

In general, (7.5) and (7.7) do not coincide. However, if we suppose that we are in a generic situation so that there are no unmoving roots, then $r_s = r_s^+ + r_s^-$ and defining $\Delta r_s = r_s^+ - r_s^-$ we find

$$z_s^\pm = \gamma\Delta r_s - 2\pi E\left[\frac{1}{2} + \frac{\gamma}{2\pi}\Delta r_s\right]. \tag{7.8}$$

8. Computation of the UV conformal weights

We shall now use the powerful machinery of the NLIE to probe the physics of the UV region of our model. Indeed, it is expected that as $L \to 0$ (the same as $T \to \infty$ in the TBA equations), the theory should flow to its UV fixed point. More precisely, the leading $1/L$ behaviour of the energy of the excited states should coincide with the results of conformal field theory (CFT), giving us an explicit expression of the central charge and all conformal weights.

According to the remarks made at the beginning of the previous section, all the results obtained in it are valid if we first send M and θ to infinity so that mL remains finite, then consider the limit $mL \to 0$. In particular we again define left/right-movers: $(r \equiv mL)$

$$\begin{aligned} \lambda_{s,k} &= \lambda^{\pm}_{s,k} \pm \log(2/r) \\ \eta_{s,k} &= \eta^{\pm}_{s,k} \pm \log(2/r) \end{aligned} \tag{8.1}$$

and the chiral counting functions:

$$Z^{\pm}_s(\lambda) = \lim_{r\to 0} Z_s(\lambda \pm \log(2/r)). \tag{8.2}$$

In the chiral limit the NLIE (6.1) becomes

$$Z^{\pm}_s(\lambda) = \pm \frac{m_s}{m} e(\pm\lambda) + \sum_{t=1}^{n} X_{st} \star Q^{\pm}_t + g^{\pm}_s(\lambda) \tag{8.3}$$

where $Q^{\pm}$ (resp. $g^{\pm}_s$) is related to Q (resp. g_s) in the obvious way.

Now we begin the computation of the finite-size corrections to the energy. We recall that

$$E = E^+ + E^- \tag{8.4}$$

with $E^{\pm} = (E \pm P)/2$. Let us choose E^+; we expand it from (4.5) and keep the dominant term in the $L \to 0$ limit:

$$E^+ = \frac{1}{L} \sum_s \frac{m_s}{m} \left[\sum_{k=1}^{M^+_{H,s}} e(\eta^+_{s,k}) - 2 \sum_{k=1}^{M^+_{S,s}} e(\sigma^+_{s,k}) - \sum_{k=1}^{M^+_{C,s}} e(\xi^+_{s,k}) - \frac{1}{2\pi} \int \mathrm{d}\lambda\, \mathrm{e}^{\lambda} Q^+_s(\lambda) \right]. \tag{8.5}$$

We have used the notation $e(\lambda)$ even for holes and special roots/holes for which one has of course $e(\lambda) = \mathrm{e}^{\lambda}$. We now use the NLIE (8.3) to eliminate the $e(\lambda)$ terms: since $Z^+_s(\eta^+_{s,k}) = 2\pi I^+_{H,s,k}$ and similar relations for special roots/holes and complex roots, we find that

$$\begin{aligned} E^+ = \frac{1}{L} \Bigg[& 2\pi (I^+_H - 2I^+_S - I^+_C) + \sum_s \Bigg[- \sum_{k=1}^{M^+_{H,s}} g^+_s(\eta^+_{s,k}) + 2 \sum_{k=1}^{M^+_{S,s}} g^+_s(\sigma^+_{s,k}) \\ & + \sum_{k=1}^{M^+_{C,s}} g^+_s(\xi^+_{s,k}) - \frac{1}{2\pi} \int \mathrm{d}\lambda\, (\mathrm{d}/\mathrm{d}\lambda) f^+_s(\lambda) Q^+_s(\lambda) \Bigg] \Bigg]. \end{aligned} \tag{8.6}$$

We have introduced the notation $f^+_s(\lambda) \equiv \frac{m_s}{m} \mathrm{e}^{\lambda} + g^+_s(\lambda)$ to recombine the differents terms where Q^+_s appears. $I^+_H \equiv \sum_s \sum_k I_{H,s,k}$, $I^+_S \equiv \sum_s \sum_k I_{S,s,k}$, $I^+_C \equiv \sum_s \sum_k I_{C,s,k}$.

Next we use a variant of the dilogarithm trick: it is the multicomponent generalization of the lemma of [6]. We state the equality

$$\begin{aligned} \sum_s \int \mathrm{d}\lambda\, (\mathrm{d}/\mathrm{d}\lambda) f^+_s(\lambda) Q^+_s(\lambda) = & -2 \sum_s \mathrm{Re} \int_{\Gamma_s} \frac{\mathrm{d}u}{u} \log(1+u) \\ & - \tfrac{1}{2} \sum_{s,t} [Q^+_s(+\infty) Q^+_t(+\infty) - Q^+_s(-\infty) Q^+_t(-\infty)] \int_{-\infty}^{+\infty} \mathrm{d}x\, X_{st}(x) \end{aligned} \tag{8.7}$$

where Γ_s is a contour in the complex plane which goes from $(-1)^{\delta_s} Z^+_s(-\infty + \mathrm{i}0)$ to $(-1)^{\delta_s} Z^+_s(+\infty + \mathrm{i}0)$ avoiding the logarithmic cut on $[-\infty, -1]$.

Using $\int_{-\infty}^{+\infty} \mathrm{d}x\, X_{st}(x) = X_{st}(\kappa = 0) = \frac{1}{\pi}\chi_{st}(+\infty) = \delta_{st} - C_{st}^{-1}/(1-\gamma/\pi)$ and computing explicitly the integral over u, we find

$$\sum_s \int \mathrm{d}\lambda\, (\mathrm{d}/\mathrm{d}\lambda) f_s^+(\lambda) Q_s^+(\lambda) = \sum_s \left(\frac{\pi^2}{6} - \frac{z_s^{+2}}{2}\right) + \tfrac{1}{2}\sum_{s,t} z_s^+ z_t^+ X_{st}(k=0)$$
$$= n\frac{\pi^2}{6} - \tfrac{1}{2}\sum_{s,t} z_s^+ z_t^+ \frac{C_{st}^{-1}}{1-\gamma/\pi}. \tag{8.8}$$

Note that no dilogarithm function is actually involved, only elementary functions appear.

The sum of all g_s^+ appearing in (8.6) simplifies enormously owing to the oddness under simultaneous exchange of $s \leftrightarrow t$ and $\lambda \leftrightarrow -\lambda$ in $\chi_{st}(\lambda)$; after some lengthy algebra we find that

$$\sum_s \left[-\sum_{k=1}^{M_{H,s}^+} g_s^+(\eta_{s,k}^+) + 2\sum_{k=1}^{M_{S,s}^+} g_s^+(\sigma_{s,k}^+) + \sum_{k=1}^{M_{C,s}^+} g_s^+(\xi_{s,k}^+)\right]$$
$$= -\sum_{s,t} \chi_{st}(+\infty)\hat{r}_s^+(r_t - \hat{r}_t^+) + 2\pi q^+ \tag{8.9}$$

where q^+ is a half-integer which depends on the number of complex roots (wide roots, roots of the first kind).

Putting everything together, the energy takes the form

$$E^+ = \frac{1}{L}\Bigg[-n\frac{\pi}{12} + 2\pi(I_H^+ - 2I_S^+ - I_C^+ + q^+)$$
$$+ \sum_{s,t} z_s^+ z_t^+ \frac{C_{st}^{-1}}{4\pi(1-\gamma/\pi)} - \pi\sum_{s,t}\hat{r}_s^+(r_t - \hat{r}_t^+)\left(\delta_{st} - \frac{C_{st}^{-1}}{1-\gamma/\pi}\right)\Bigg]. \tag{8.10}$$

Using the expression (7.5) for z_s^+ and performing some recombinations, we can write the final result

$$E^\pm = \frac{2\pi}{L}\left(-\frac{c}{24} + \Delta^\pm + p^\pm\right) \tag{8.11}$$

where

$$c = n \tag{8.12}$$

is the central charge,

$$\Delta^\pm = \frac{\sum_{s,t} C_{st}^{-1}[r_s + (1-\gamma/\pi)(2r_s^\pm - r_s)][r_t + (1-\gamma/\pi)(2r_t^\pm - r_s)]}{8(1-\gamma/\pi)} \tag{8.13}$$

are the (primary) conformal weights, and

$$p^\pm = \pm(I_H^+ - 2I_S^+ - I_C^+ + q^+) - \tfrac{1}{2}\sum_s \hat{r}_s^\pm(r_s - \hat{r}_s^\pm + M_s - 2M_s^\pm) \tag{8.14}$$

is a half-integer. In view of (8.11) one can reasonably assume that $p^\pm$ is in fact non-negative, which can be checked directly.

We now exclude special configurations with unmoving roots, so that $r_s = r_s^+ + r_s^-$, and $2r_s^\pm - r_s = \pm\Delta r_s$ with, as before, $\Delta r_s \equiv r_s^+ - r_s^-$†. This slightly simplifies the form of (8.13), and allows the following interpretation: the central charge (8.12) indicates n free bosons. In fact general arguments (see appendix A) show that the UV fixed point of the affine

† Special attention must be paid to the case $\Delta r_s = 0$, in which, to avoid an unmoving root at $Z = 0$ (cf equation (7.8)), one must choose the appropriate value of M_s mod $2h$ so that $\delta_s = 0$.

Toda with imaginary coupling should be a multicomponent Coulomb gas (i.e. compactified free bosons). Indeed, the conformal weights (8.13) are closely related to those of n free bosons, as is shown in appendix A. For example, in the A_1 case, they are connected with the deformed chiral Gross–Neveu model, whose bosonization is the sine-Gordon model. The UV conformal weights (8.13) are related to the IR conformal weights of the corresponding spin chain [16] by exchange of r_s and Δr_s.

It should be pointed out that the finite-size correction to the energy *does not depend* on the actual values of the rapidities of holes and complex roots: it only depends on their number, or more precisely of the partial (chiral) $\mathfrak{g}_0$ quantum numbers. In particular this indicates that the string hypothesis, which constrains the positions of the complex roots, is useless here. Indeed we have not made any use of it, knowing that for low-lying excited states it is in fact violated.

9. Twist and quantum group truncation

In the sine-Gordon model, it is known that at rational values of γ/π, one can consistently restrict the theory to a smaller Hilbert [17, 18] which in particular displays a different UV behaviour, reproducing the minimal models. We shall show that such a truncation can be extended to the complex affine Toda model.

The key ingredient of the truncation is the quantum group symmetry and its representation theory [19, 18]. Since the representation theory of $U_q(\mathfrak{g})$ is well developed and closely resembles that of $U_q(\mathfrak{sl}(2))$, we expect no particular difficulty. However, implementing the truncation in the Bethe ansatz framework raises several questions.

The natural way to implement the truncation is to introduce a twist in the Bethe ansatz equations. Indeed it is known that Bethe ansatz equations with twist [20] are related to restricted solid-on-solid (RSOS) models, which themselves are equivalent to restricted sine-Gordon (at least in the UV limit), but this is a rather indirect connection, and we would like to have a more direct derivation of the truncation. The second problem is specific to the NLIE approach: as we are considering the theory on a compactified space of length L, it *does not* possess the quantum group $U_q(\mathfrak{g})$ symmetry. To summarize, even in the $U_q(\mathfrak{sl}(2))$ case, in which the twist in the NLIE equations has been done [21], it has not been justified that this procedure was the same as the quantum group truncation discussed earlier. We shall now give such a justification for the generalized case of affine Toda. The quantum group symmetry will reappear after a modular transformation which we are naturally led to doing. In the UV limit we shall find results which bear the same connection to the Jimbo–Miwa–Okado models [22] as restricted sine-Gordon to the RSOS models.

9.1. The group-theoretic background

Let us remind the reader that the affine Toda model (with imaginary coupling constant) associated with the simply laced Lie algebra $\mathfrak{g}$ consists of n bosonic fields, grouped into a field ϕ which belongs to the Cartan subalgebra $\mathfrak{g}_0$. The action is given by

$$\mathcal{S} = \frac{1}{\beta^2}\int \mathrm{d}^2x \left[(\partial_\mu\phi)^2 + m^2\sum_{s=0}^{n}\exp(-\mathrm{i}\langle\alpha_s,\phi\rangle)\right]. \tag{9.1}$$

The α_s, $s = 0\ldots n$ are the simple roots of $\hat{\mathfrak{g}}$ (alternatively one can consider that the α_s, $s = 1\ldots n$ are the simple roots of $\mathfrak{g}$, and $-\alpha_0$ is the highest root of $\mathfrak{g}$).

Since $\mathfrak{g}_0$ possesses a scalar product, we identify it with its dual space (weight space). With this convention, one can decompose ϕ in the basis of fundamental weights:

$\phi = \sum_s \phi_s w_s$. Using the orthogonality relations $\langle \alpha_s, w_t \rangle = \delta_{st}$ $(s, t = 1 \ldots n)$, it is obvious to check that this model has a $\mathbb{Z}^n$ symmetry, with generators $T_s : \phi_s \to \phi_s + 2\pi$. In order to select the eigenvalues of the T_s, one introduces the 'shifted' partition function† $\boldsymbol{Z}_k$: $(k = (k_1, \ldots, k_n) \in \mathbb{Z}^n)$

$$\boldsymbol{Z}_k \equiv \mathrm{tr}(\exp(-\beta H_L) T_1^{k_1} \ldots T_n^{k_n}) = \int_{\substack{\phi(t=\beta,x)=\phi(t=0,x)+2\pi k \\ \phi(t,x=L)=\phi(t,x=0) \bmod 2\pi}} [\mathrm{d}\phi] \mathrm{e}^{-\mathcal{S}[\phi]}. \tag{9.2}$$

We have considered the model on a finite space of size L and have imposed periodic boundary conditions *modulo* 2π *only* for the ϕ_s. H_L is the Hamiltonian in the corresponding operator formalism. We have also taken a finite-temperature β (this $\beta = 1/T$ has nothing to do, of course, with the constant in front of the action (9.1)); later, when we are concerned with the ground state and low-lying excited states only, we shall take the limit $\beta \to \infty$, these states correspond to the first terms in the large β expansion.

Next we introduce the partition function restricted to the sector of the Hilbert space of the Toda, in which the T_s have the eigenvalues $\mathrm{e}^{\mathrm{i}\omega_s}$:

$$\begin{aligned} \boldsymbol{Z}(\Omega) &\equiv \mathrm{tr}_\Omega(\exp(-\beta H_L)) \\ &= \sum_{k_1,\ldots,k_n=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\sum_{s=1}^n \omega_s k_s} \boldsymbol{Z}_k. \end{aligned} \tag{9.3}$$

The eigenvalues are parametrized by $\Omega \in \exp(\mathrm{i}\mathfrak{g}_0)$: $\Omega = \exp(\mathrm{i}\omega)$ where $\omega = (\omega_1, \ldots, \omega_n)$ in the basis of fundamental weights (after identifying, as above, $\mathfrak{g}_0$ and weight space). tr_Ω means the trace in the sector $T_s = \mathrm{e}^{\mathrm{i}\omega_s}$.

In order to understand why these subtleties are usually neglected, let us first consider the $L \to \infty$ (infinite space) limit: then the transition (in time) between different classical vacua is suppressed and $Z_k \to 0$ for $k \neq 0$; $\boldsymbol{Z}(\Omega)$ becomes independent of Ω i.e. all the sectors of the Toda become degenerate.

The functional integral (9.3) can receive another interpretation by exchanging the roles of space and time; after this modular transformation, the operatorial interpretation becomes

$$\boldsymbol{Z}(\Omega) = \mathrm{tr}_1(\exp(-L H_\beta)\Omega) \tag{9.4}$$

(tr_1 means the trace over the trivial sector of the $\mathbb{Z}^n$ symmetry, i.e. $\phi_s \equiv \phi_s + 2\pi$) with Ω considered as the exponential of an element of the Cartan algebra $\mathfrak{g}_0$ (acting on the whole Hilbert space). This formula can be guessed by noticing that after the modular transformation, the numbers k_s precisely describe the topological charges which are associated with the $\mathfrak{g}_0$ symmetry. We shall call $\boldsymbol{Z}(\Omega)$ the twisted partition function since both in the transfer matrix language (see next paragraph) or in a 'fermionized' language (using boson–fermion equivalence in 2D; though this introduces additional subtleties owing to fermionic boundary conditions and modular invariance that we do not wish to discuss) Ω appears as a twist in the spatial boundary conditions.

Let us now consider the limit $\beta \to \infty$. In this limit, H_β should commute with the action of the full quantum group $U_q(\mathfrak{g})$, enlarging the $\mathfrak{g}_0$ symmetry. Then one can decompose the Hilbert space according to $U_q(\mathfrak{g})$ representations, and use a character expansion‡:

$$\boldsymbol{Z}(\Omega) = \sum_R \chi_R(\Omega) \boldsymbol{Z}_R \tag{9.5}$$

† A more standard denomination would be 'twisted' partition function, but we reserve the word 'twisted' for a slightly different, in fact dual, situation, cf (9.3).

‡ Note that decomposition (9.5) is not the same decomposition as (9.3): in (9.5) the sum is over all *highest* weights of $U_q(\mathfrak{g})$ whereas in (9.3) it is over all (integral) weights.

where R runs over all highest weight representations of $U_q(\mathfrak{g})$, and $\boldsymbol{Z}_R$ is the partition function of the sector of the Hilbert space with representation R (divided by the dimension of the representation).

So far, in all the previous sections of this paper we have implicitly chosen $\Omega = 1$: this also corresponds, from what has been said, to the 'trivial' sector of the Toda for the $\mathbb{Z}^n$ symmetry. We shall now choose a non-trivial Ω which selects 'good' representations of $U_q(\mathfrak{g})$ for γ/π rational. More precisely, we choose

$$\Omega = q^{2H} \equiv q^{\sum_{\alpha>0} H_\alpha} \tag{9.6}$$

where H_α is the element of the Cartan algebra $\mathfrak{g}_0$ associated with the positive root α. This corresponds more explicitly with $\omega_s = 2\gamma$†. Then it is known that $\chi_R(\Omega) = 0$ for all the 'bad' representations (indecomposable but not irreducible representations, and a few others), and, according to (9.5), we are left with contributions from the 'good' representations, with prefactors $\chi_R(\Omega)$ which correctly account for the truncation of the tensor product (for a more thorough analysis in the A_1 case see [18, 19]).

It is worth stressing that *any* value of Ω *is a priori* conceivable: the spectrum of the generators T_s is the whole $U(1)$ circle (contrary to what has been written in the recent literature). This does not contradict the quantum group truncation, because one should be careful that the truncation takes place when the *time* direction is compactified with length L, i.e. after a modular transformation has been performed. In particular the 'ground state' contribution for the twisted Toda does not correspond at all to the ground-state contribution in this dual picture: in contrast we are considering the theory at finite temperature $T = 1/L$ (which we eventually send to infinity when we look at the UV region). So the finite-size correction varies continuously with the twist Ω, as we shall see in the next paragraph, but only for particular (discrete) values does it have an interpretation in terms of a truncated Hilbert space.

9.2. Twist and Bethe ansatz

It is particularly simple to add a twist in our formalism: the twisted version of the transfer matrix (2.2) is

$$T(\Lambda, \Theta, \Omega) = \mathrm{tr}_{\mathrm{aux}}[L_1(\Lambda - \mathrm{i}\Theta)L_2(\Lambda + \mathrm{i}\Theta)\ldots L_{2M-1}(\Lambda - \mathrm{i}\Theta)L_{2M}(\Lambda + \mathrm{i}\Theta)\Omega] \tag{9.7}$$

Ω acts in the auxiliary space. In the scaling limit, one can easily convince oneself that the twisted transfer matrix leads to the model described by the partition function $\boldsymbol{Z}(\Omega)$ of (9.3).

Of course, the addition of the twist preserves the integrability; to diagonalize T we now have twisted Bethe ansatz equations:

$$\prod_{j=1}^{M_s} \frac{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \lambda_{s,j}) + \mathrm{i}))}{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \lambda_{s,j}) - \mathrm{i}))} \prod_{t|\langle st\rangle} \prod_{j=1}^{M_t} \frac{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \lambda_{t,j}) - \mathrm{i}/2))}{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \lambda_{s,j}) + \mathrm{i}/2))}$$
$$= \mathrm{e}^{\mathrm{i}\omega_s} \left[\frac{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \theta) + \mathrm{i}/2))}{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} - \theta) - \mathrm{i}/2))} \frac{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} + \theta) + \mathrm{i}/2))}{\sinh(\gamma(\frac{h}{2\pi}(\lambda_{s,k} + \theta) - \mathrm{i}/2))} \right]^{M\delta_{s1}}. \tag{9.8}$$

Finally, this introduces an extra term ω_s in the definition of the counting function Z_s, and the NLIE (3.17) becomes

$$Z_s = m_s L \sinh\lambda + \sum_{t=1}^{n} \left[\frac{C_{st}^{-1}}{1 - \gamma/\pi}\omega_t + X_{st} \star Q_t + \sum_{k=1}^{M_{H,t}} \chi_{st}(\lambda - \eta_{t,k})\right.$$

† The factor of 2 originally comes from our convention for the definition of the deformation parameter q; other authors use $q' \equiv q^2$, which removes this 2.

$$-2\sum_{k=1}^{M_{S,t}}\chi_{st}(\lambda-\sigma_{t,k})-\sum_{k=1}^{M_{C,t}}\chi_{st}(\lambda-\xi_{t,k})\Bigg]. \tag{9.9}$$

9.3. The UV limit of the truncated theory

One can again probe the UV fixed point (of the truncated theory) by sending L to 0. This amounts to redoing the calculations of section 8 in the presence of the twist. We shall only rewrite the relations that are modified in the process. Equation (7.5) becomes

$$z_s^+=\gamma(2r_s^+-r_s)+2\pi(\hat r_s^+-r_s^+)+\omega_s. \tag{9.10}$$

When going from (8.5) to (8.6) one uses the NLIE, so one gains an extra term:

$$E^+=\frac{1}{L}\Bigg[2\pi(I_H^+-2I_S^+-I_C^+)+\sum_s\Bigg[-\sum_{k=1}^{M_{H,s}^+}g_s^+(\eta_{s,k}^+)+2\sum_{k=1}^{M_{S,s}^+}g_s^+(\sigma_{s,k}^+)\\ +\sum_{k=1}^{M_{C,s}^+}g_s^+(\xi_{s,k}^+)-\hat r_s^+\sum_t\frac{C_{st}^{-1}}{1-\gamma/\pi}\omega_t-\frac{1}{2\pi}\int \mathrm{d}\lambda\,(\mathrm{d}/\mathrm{d}\lambda)\,f_s^+(\lambda)Q_s^+(\lambda)\Bigg]\Bigg]. \tag{9.11}$$

Finally $E^\pm$ is given by

$$E^\pm=\frac{2\pi}{L}\Bigg[-\frac{n}{24}+p^\pm\\ +\frac{\sum_{s,t}C_{st}^{-1}[r_s\pm(1-\gamma/\pi)\Delta r_s\mp\omega_s/\pi][r_t\pm(1-\gamma/\pi)\Delta r_t\mp\omega_t/\pi]}{8(1-\gamma/\pi)}\Bigg] \tag{9.12}$$

where $p^\pm$ is unchanged. Note that this expression, just like (8.13), correctly behaves under space parity: E^+ and E^- (or Δ^+ and Δ^-) are exchanged by $r_s^\pm\leftrightarrow -r_s^\mp$ (the ω_s, from their definition, are unaffected by space parity).

For generic values of γ and of the ω_s this formula simply gives the finite-size corrections of affine Toda in a sector $T_s=\mathrm{e}^{\mathrm{i}\omega_s}$. In particular, one finds that the true ground state of the theory is in the sector $\Omega=1$, since for $r_s=0$ the energy increases as Ω moves away from 1.

As explained in previous paragraph, the result (9.12) acquires a new significance for γ/π rational and Ω fixed by (9.6). The new central charge of the truncated theory is smaller than n, since the second line of (9.12) is no longer purely quadratic in the r_s, $r_s^\pm$ (it has a constant and a linear part).

Let us first consider $\gamma=\pi/(p+1)$. Setting $\omega_s=2\gamma$, one finds the result (using the strange formula $12\sum_{s,t}C_{st}^{-1}=h\dim\mathfrak{g}$)

$$\begin{cases} c=n\left(1-\dfrac{h(h+1)}{p(p+1)}\right)\\ \Delta^\pm=\Delta_0+\dfrac{\sum_{s,t}C_{st}^{-1}[m_s(p+1)\pm n_sp\mp 1][m_t(p+1)\pm n_tp\mp 1]}{2p(p+1)}\end{cases} \tag{9.13}$$

where $\Delta_0\equiv(c-n)/24=-\frac{n}{24}\frac{h(h+1)}{p(p+1)}$, and $r_s=2m_s$, $\Delta r_s=2n_s$. (9.13) is characteristic of a representation of the $W(\mathfrak{g})$ extended conformal algebra corresponding to unitary RCFTs [23]. For $\mathfrak{g}=A_1$ and $n_1=0$ (9.13) is equivalent to what was found in [21].

Let us now suppose that $\gamma = \pi(q-p)/q$ (p and q coprime integers). Upon replacement of γ and $\omega_s = 2\gamma$ with their values one finds

$$c = n\left\{\begin{aligned}&\left(1 - h(h+1)\frac{(p-q)^2}{pq}\right)\\ &\Delta^{\pm} = \Delta_0 + \frac{\sum_{s,t} C_{st}^{-1}[m_s q \pm n_s p \mp (q-p)][m_t q \pm n_t p \mp (q-p)]}{2pq}\end{aligned}\right. \tag{9.14}$$

with similar notations as in (9.13); $\Delta_0 = -\frac{n}{24}\frac{h(h+1)(p-q)^2}{pq}$. This time we find the representations of $W(\mathfrak{g})$ corresponding to all RCFTs (p, q).

10. Conclusion and prospects

We have presented here some results concerning the affine Toda model associated with a simply laced Lie algebra. We have written NLIE which allow us to interpolate excited states from $L = \infty$ (IR region) to $L = 0$ (UV region). The two limits have been discussed. In the UV region we recover results of CFT. One should study more thoroughly the $L \to \infty$ limit in the attractive regime: it would give the full mass spectrum and scattering of the theory. This promises to be a rather complex task, because of the problem of classifying the breathers.

Finally, the quantum group truncation has been described in detail, and the corresponding NLIE written. We have checked that the truncated theory does display a central charge and conformal weights which are compatible with $W(\mathfrak{g})$ symmetry. However, this requires some further clarification: indeed it is not completely obvious which primary operators are present, and under the form of which states. For example, in the $\mathfrak{g} = A_1$ case, this is probably related to the subtle differences which exist between the various 'equivalent' formulations of the model (cf appendix A). A similar analysis is probably possible for a general algebra $\mathfrak{g}$, but it has not yet been performed.

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Appendix A. Multicomponent Coulomb gas

There are many equivalent ways of introducing the multicomponent generalization of the conformal Coulomb gas. The most appropriate one for us is to start from action (9.1); in the UV limit it can be shown that the mass term (after appropriate renormalization), for $\beta^2 < 8\pi$, tends to zero. Rewriting the remainder of the action in terms of the rescaled components $\Phi_s = R\phi_s$ with $R \equiv \sqrt{4\pi}/\beta$ results in

$$\mathcal{S}^{\text{Coulomb}} = \frac{1}{4\pi}\int \mathrm{d}^2x \sum_{s,t=1}^{n} C_{st}^{-1}(\partial_\mu\Phi_s)(\partial_\mu\Phi_t) \tag{A.1}$$

(the normalization of the action is conventional: it fixes the radius of compactification, which we have chosen for $n = 1$ as in [24]). From the discussion of section 9.1 it should also be clear that in the trivial sector of the $\mathbb{Z}^n$ symmetry, one should identify Φ_s and $\Phi_s + 2\pi R$ which means we are dealing with compactified free bosons on circles of radius R (but not n independent compactified bosons).

There are additionnal symmetries arising in this UV limit: besides the topological currents $\epsilon_{\mu\nu}\partial_\nu\Phi_s$ associated with our usual $\mathfrak{g}_0 = \mathfrak{u}(1)^n$ symmetry, we have the obvious currents $\partial_\mu\Phi_s$ associated with $\Phi_s \to \Phi_s +$ constant (the corresponding symmetry group is $U(1)^n$ since $\Phi_s \equiv \Phi_s + 2\pi R$). We now have two sets of quantum numbers describing a state: the $\mathfrak{g}_0$ quantum numbers m_s (also called winding numbers or magnetic charges) and the e_s ('target space' momenta in the string picture, or electric charges). Standard arguments [25] allow us to find the full spectrum. The primary (for the $U(1)^n \times U(1)^n$ Kac-Moody algebra) conformal weights are given by

$$\Delta^\pm = \frac{1}{4}\sum_{s,t} C^{-1}_{st}\left(\sum_{s'} C_{ss'}e_{s'}/R \pm m_s R\right)\left(\sum_{t'} C_{tt'}e_{t'}/R \pm m_t R\right) \tag{A.2}$$

where the m_s and the e_s are the aforementioned quantum numbers (integers).

Note in particular that the purely electric operators $e^{i\langle\alpha,\phi\rangle}$ have dimension

$$\Delta^\pm = \frac{1}{4}\frac{\langle\alpha,\alpha\rangle}{R^2} \tag{A.3}$$

so that for the perturbing operators of (9.1) we have $\Delta^\pm = \frac{1}{2}R^2 = \beta^2/8\pi$. They are relevant for $\beta^2 < 8\pi$, as expected.

Naively, there are several ways of matching the conformal weights (8.13) and (A.2), owing to the many partial dualities relating different radii of compactification. One finds that the correct relation to impose is $\gamma = \pi - \beta^2/8$, so that R is given by

$$R = \frac{1}{\sqrt{2(1-\gamma/\pi)}} \tag{A.4}$$

and the identifications are

$$r_s = m_s \tag{A.5a}$$

$$\Delta r_s = 2\sum_{t=1}^{n} C_{st}e_t \tag{A.5b}$$

where $\Delta r_s = r_s^+ - r_s^- = \sum_t C_{st}(M_t^- - M_t^+)$ so that $e_s = \frac{1}{2}(M_s^- - M_s^+)$.

The first identification (A.5a) a was expected on general grounds. Note that owing to its definition (2.6), the r_s span only a subset of the integer lattice. One first constraint is that all r_s are positive; this is due to the fact that we are only considering highest weight states. If we also considered lower weight states (e.g. antisolitons and not just solitons for sine-Gordon), it is expected that we would recover negative values. Furthermore, the r_s are always in a sublattice: for example, for $\mathfrak{g} = A_n$, one easily finds that the r_s satisfy the constraint $\sum_s s r_s \equiv 2M \bmod n+1$ (conservation of the number of boxes of the Young tableau mod$n+1$). However, as is usual in the Bethe ansatz, when M is sent to infinity one can consider all values of $2M \bmod n+1$ simultaneously (possibly considering an odd number of sites), so that one recovers all possible r_s.

Let us now discuss briefly the allowed values of e_s: it would seem that the e_s can be half-integers (in fact, extrapolating (A.5b) to arbitrary values of Δr_s, one would even find that $e_s \in \frac{1}{2h}\mathbb{Z}$). The situation is particularly clear in the $\mathfrak{g} = A_1$ case, in which $m = r$ and $e = \frac{1}{4}\Delta r$. The correct interpretation of this non-integerness is that the Bethe ansatz model we are considering does not describe sine-Gordon, but really an equivalent model: the deformed $SU(2)$ chiral Gross–Neveu model [26]; note that this is not the same deformation as the one introduced in [8]), in which physical excitations have electric charge $\pm\frac{1}{4}$. This model should be distinguished from the two other 'equivalent' models: the sine-Gordon model itself, in which electric charges are integer; and the massive Thirring model, in which they are half-integer [27] (note that we use different conventions for the radius and the electric and magnetic charges from [27]). Deformations of the chiral Gross–Neveu model have central charge $c = 2$, so one must first remove a decoupled $c = 1$ massless sector (the separation of the sectors destroys the modular properties of the remaining $c = 1$ model, which is why the deformed Gross–Neveu model was not found in [27] starting from modular invariance considerations).

Let us dispell a possible confusion by noting that (still in the $\mathfrak{g} = A_1$ case), if we restrict the theory to an even number of solitons by keeping the number of sites $2M$ even, that is if both r and Δr are even, we may also identify directly the spectrum (8.13) with (A.2) by setting $m = r/2$, $e = \Delta r/2$ and the radius $R' = 2R$ (or, using the *exact* electromagnetic duality of the $c = 1$ compactified boson, $e = r/2$, $m = \Delta r/2$ and $R'' = 1/R$). Then electric and magnetic charges are integer. However, this point of view has several drawbacks. The problem stems from the fact that the perturbing operator is now different: it is $\cos(2\Phi/R')$ and not $\cos(\Phi/R')$. This implies that, with the compactification $\Phi \equiv \Phi + 2\pi R'$, the potential has *two* minima instead of one, and it is natural to consider that magnetic charges are half-integers. In particular, the elementary physical excitations (solitons) have magnetic charge $\frac{1}{2}$.

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